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A Mathematical Model for a Set of Microsystems[1]

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Starting with the usual Hilbert space formulation of quantum mechanics we construct a mathematical model which proves that the set of axioms for the theoretical description of single microsystems developed by Ludwig is free of contradictions and admits of nontrivial solutions.

1. INTRODUCTION

Since the discovery of the statistical interpretation of quantum mechanics several attempts were made to include the description of single quantum systems into the mathematical picture of quantum theory. [Cf. Everett (1957), Piron (1972), Ludwig (1975, 1977), and Davidon (1977).] These attempts were made to discuss or even "solve" epistemological problems of quantum mechanics.

In the present paper we are concerned with Ludwig's approach to this problem (1975, 1977). Ludwig developed a pretheory to quantum mechanics in which, analogous to the sample space of Kolmogorov's probability theory, a set of microsystems is introduced. Ludwig understands this pretheory as a tool to bridge the gap between experiments and the usual formalism of quantum mechanics. Moreover, it is used to discuss the misunderstandings and paradoxes that arise in the interpretation of quantum mechanics on a mathematical footing.

Since this pretheory is introduced axiomatically the question arises whether the set of axioms admits of nontrivial solutions, i.e., whether the probability functions considered in this pretheory are compatible with Hilbert space models of dimension greater than 1.

[1] Dedicated to Professor Günther Ludwig on the occasion of his 60th birthday.

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To construct such a model for the pretheory in question we consider an irreducible quantum mechanical system, i.e., a system without superselection rules, which is described in an infinite dimensional separable Hilbert space H by the set of ensembles

$$K = \{W \in B(H)/W^* = W, W \geqslant 0, \mathrm{Tr}\, W = 1\}$$

and the set of effects

$$L = \{F \in B(H)/F^* = F, 0 \leqslant F \leqslant 1\}$$

Tr (WF) is the probability of measuring the effect F in the ensemble W.

In the sequel we construct a countable set M of "microsystems" equipped with a structure $(\mathbf{Q}, \mathbf{R}_0, \mathbf{R})$, where $\mathbf{Q} \subset \mathscr{P}(M)$ is a system of preparing procedures and $\mathbf{R}_0 \subset \mathscr{P}(M)$ and $\mathbf{R} \subset \mathscr{P}(M)$ are systems of recording methods and procedures, respectively. In addition we specify mappings

$$\varphi\colon \mathbf{Q}' \to K \qquad \text{where} \qquad \mathbf{Q}' = \mathbf{Q}\backslash\{\varnothing\} \text{ and } \psi\colon \mathscr{F} \to L$$

such that the axioms postulated by Ludwig are fulfilled. [$\mathscr{F} = \{(b_0, b)/b_0 \in \mathbf{R}_0, b \in \mathbf{R}, b_0 \supset b\}$ is the set of effect procedures.]

Though M equipped with this structure could be called a set of microsystems, in fact, only if the preparing and effect procedures are connected with the reality by mapping principles will we use the expression "microsystems," preparing and recording microsystems in this mathematical model. Since we do not intend to repeat the definitions and semantics of Ludwig's description of single microsystems, we must hope that the usage of these physical expressions together with the mathematical construction of the model will also be sufficient to elucidate the main features of Ludwig's scheme. The mathematical model carries realistic features insofar as it fulfills the axioms which are postulated for the mathematical picture.

2. CONSTRUCTION OF THE SET OF MICROSYSTEMS

At the beginning we select a countable set $L_c \subset L$ of effects as images of effect procedures under the mapping ψ to be constructed. ψ has to be chosen such that all observables can be approximated in a certain sense by observables consisting of effects of L_c. In view of this approximation procedure of Ludwig (in preparation), it is sufficient to consider finite observables. A finite observable is a finite Boolean algebra Σ endowed with an effective L-valued measure which assumes the value $1 \in L$ on the unit element of the Boolean algebra. Every finite Boolean algebra is atomic and is determined uniquely up to isomorphism by the number n of its atoms $\sigma_1, \ldots, \sigma_n$. The L-valued measure F satisfies $\sum_{i=1}^{n} F(\sigma_i) = 1$. On the other hand, for each finite subset $\{F_i\}_{i=1,\ldots,n} \subset L$ with $F_i \neq 0$, $\sum_{i=1}^{n} F_i = 1$, there is a Boolean algebra Σ_n, unique up to isomorphism, endowed with an effective L-valued measure which assumes the values $F_1, \ldots, F_n$ on the set of atoms of Σ_n. Hence a finite observable can be characterized by a subset $S = \{F_i\}_{i=1,\ldots,n} \subset L$ with $\sum_{i=1}^n F_i = 1$. ($F_i = F_k$ for $i \neq k$ is admitted.) Let M_o be the set of all such subsets of L.

Lemma 1: There is a countable subset $L_c \subset L$ such that for $S \in M_o$ and a σ^*-neighborhood U of 0 there is $S' \in M_o$ with $S' \subset L_c$ and $F_i - F_i' \in U$ for all $i = 1, \ldots, n$ ($F_i \in S$, $F_i' \in S'$).

Of course, it can be assumed that $F = \sum_{i\in I} F_i \in L_c$ for $I \subset \{1, \ldots, n\}$ if $F_1, \ldots, F_n \in L_c$ and $\sum_{i=1}^n F_i = 1$.

The proof of this lemma is given in the Appendix.

Now, let $M_{oc} = \{S \in M_o / S \subset L_f\}$. For $S = \{F_1, \ldots, F_n\}$ we consider a representation of the Boolean algebra Σ_S with n atoms in N such that every atom is an infinite subset of N. (For example, $\sigma_i = \{i + (m - 1)n/m \in \mathsf{N}\}$ for the atoms σ_i of Σ_S.) A pair (S, σ), with $\sigma \in \Sigma_S$, $\sigma = \bigcup_{i\in I} \sigma_i$, $I \subset \{1, \ldots, n\}$, and $\sigma_1, \ldots, \sigma_n$ being the atoms of Σ_S, determines an effect

$$F(S, \sigma) = \sum_{i\in I} F_i \tag{2.1}$$

For S fixed the mapping $\Sigma_S \ni \sigma \mapsto F(S, \sigma)$ is an observable which assumes the values $F_i \in S$ on the set of atoms of Σ_S.

The discussion of recording methods and procedures that correspond to these observables will be postponed after the introduction of the set of preparing procedures.

We choose a countable dense subset $K_c = \{W_\nu\}_{\nu=1,\ldots} \subset K$ of ensembles as images of preparing procedures under the mapping φ to be constructed. We consider finite sets of pairs $\{(\nu_1, \lambda_1), \ldots, (\nu_n, \lambda_n)\}$, $\nu_i \in \mathsf{N}$, λ_i, rational numbers of the semiclosed interval $(0, 1]$ with $\sum_{i=1}^n \lambda_n = 1$. The family of these finite sets may be denoted by M_P, where M_P is denumerable. $T = \{(\nu_1, \lambda_1), \ldots, (\nu_n, \lambda_n)\} \in M_P$ characterizes a so-called preparator, i.e., a preparing apparatus equipped with signals $i = 1, \ldots, n$ never occurring simultaneously such that micro-objects prepared by T can be selected according to the signals. The signal i occurs with the probability λ_i and the correspondingly selected micro-objects are described by the ensemble W_{ν_i}. Hence a pair (T, τ), $T \in M_P$, $\tau \subset \{1, \ldots, n_T\}$, characterizes a preparing procedure in which the micro-objects of the preparator T are selected according to the occurrence of the signals $\tau \subset \{1, \ldots, n_T\}$, and the corresponding ensemble is

$$W(T, \tau) = \left(\sum_{i\in\tau} \lambda_i\right)^{-1} \sum_{i\in\tau} \lambda_i W_{\nu_i} \qquad \text{for } \tau \neq \phi \tag{2.2}$$

Now, given $\nu \in \mathsf{N}$ and $S \in M_{oc}$ there is a greatest element $\sigma(\nu, S) \in \Sigma_S$ such that

Tr $[W_\nu F(S, \sigma(\nu, S))] = 0$. Since $F(S, \mathsf{N}) = 1$ we have $\sigma(\nu, S) \neq \mathsf{N}$. The countable set M of microsystems is defined

$$M = \bigcup_{\substack{T \in M_p \\ i \in \{1,\dots,n_T\} \\ S \in M_{oc}}} [\{(T, i, S)\} \times \sigma^*(\nu_i, S)] \tag{2.3}$$

where * denotes the complementation in N and the index ν_i is determined by $T \subset \{(\nu_1, \lambda_1), \dots, (\nu_{n_T}, \lambda_{n_T})\}$.

M is a subset of $M_P \times \mathsf{N} \times M_{0_c} \mathsf{W}\, \mathsf{N}$. According to the above considerations the first two factors label the preparing procedure by which the micro-object considered is prepared. The third factor labels the recording method and the last one the recording procedure applied to the micro-object. This will be explained more explicitly in Section 3.

By the introduction of $\sigma^*(\nu, S)$ we allow sets of microsystems to be empty if the corresponding probabilities are zero.

3. THE CONSTRUCTION OF PREPARING AND RECORDING PROCEDURES AND THE DISCUSSION OF LUDWIG'S AXIOMS

Proceeding along the ideas outlined at the end of Section 2 we define the set of preparing procedures as follows:

$$\mathbf{Q} = \left\{ a_{T,\tau} = \bigcup_{\substack{i \in \tau \\ S \in M_{0_c}}} [\{(T, i, S)\} \times \sigma^*(\nu_i, S)] \middle/ T = \{(\nu_1, \lambda_1), \dots, (\nu_{n_T}, \lambda_{n_T})\},\ \tau \subset \{1, \dots, n_T\} \right\}$$

and $\varphi(a_{T,\tau}) = W(T, \tau)$ for $a_{T,\tau} \neq \varnothing$.

$\mathbf{Q}$ is a system of selection procedures [see Ludwig (1975, 1977)] since for $T \neq T'$ we have $a_{T,\tau} \cap a_{T',\tau'} = \varnothing$ and $a_{T,\tau} \cap a_{T,\tau'} = a_{T,\tau\cap\tau'}$ and $a_{T,\tau} \backslash a_{T,\tau'} = a_{T,\tau\backslash\tau'}$ for $\tau' \subset \tau$. For $T = \{(\nu_1, \lambda_1), \dots, (\nu_n, \lambda_n)\}$ and $\tau, \tau' \subset \{1, \dots, n\}$ with $\tau \supset \tau'$ we define $\lambda_Q(a_{T,\tau}, a_{T,\tau'}) = (\sum_{\nu\in\tau'} \lambda_\nu)(\sum_{\nu\in\tau} \lambda_\nu)^{-1}$ for $\tau' \neq \varnothing$. $\lambda_Q(a_{T,\tau}, a_{T,\tau'})$ fulfills all axioms for a relative probability [AS 1 of Ludwig (1977)] and thus $\mathbf{Q}$ endowed with λ_Q is a system of statistical selection procedures [Axiom APS 1 of Ludwig (1977)].

A direct mixture of preparing procedures a' and a'' corresponds to a preparing apparatus which mixes by chance micro-objects prepared by apparatus corresponding to the procedures a' and a''. For $a_{T',\tau'}, a_{T'',\tau''} \in \mathbf{Q}$, and a rational number $\alpha \in (0, 1)$, there is a direct mixture $a_{T,\tau} \in \mathbf{Q}$ of $a_{T,'\tau'}$ and

$a_{T'',\tau''}$ with a ratio α. Namely, let

$$T = \{(\nu_1, \lambda_1), \ldots, (\nu_{n'+n''}, \lambda_{n'+n''})\}$$
$$= \{(\nu'_1, \alpha\lambda'_1), \ldots, (\nu'_{n'}, \alpha\lambda'_{n'}), (\nu'_1, (1-\alpha)\lambda''_1), \ldots, (''_{m''}, (1-\alpha)\lambda''_{m''})\}$$

and let τ be the set of indices of $\{1, \ldots, n' + n''\}$ corresponding to $\tau' \cup \tau''$. We have $\varphi(a_{T,\tau}) = \alpha\varphi(a_{T',\tau'}) + (1-\alpha)\varphi(a_{T'',\tau''})$.

We shall now discuss the recording of micro-objects. For an observable $S \in M_{oc}$ we consider a recording method

$$b_{0S} = \bigcup_{\substack{T \in M_p \\ i \in (1, \ldots, n_T)}} [\{(T, i, S)\} \times \sigma^*(\nu_i, S)]$$

In addition to the set R_0 of all these recording methods we define the set of recording procedures

$$\mathbf{R} = \left\{ b_{S,\sigma} = \bigcup_{\substack{T \in M_p \\ i \in \{1, \ldots, n_T\}}} [\{(T, i, S)\} \times (\sigma \cap \sigma^*(\nu_i, S))]/S \in M_{oc}, \sigma \in \Sigma_S \right\}$$

Hence in this model an effect procedure $f = (b_{0S}, b_{S,\sigma})$ is determined by an observable $S \in M_{oc}$ and $\sigma \in \Sigma_S$. Let

$$\psi(b_{0S}, b_{S,\sigma}) = F(S, \sigma)$$

[see equation (2.1)].

$\mathbf{R}_0$ is a trivial system of statistical selection procedures such that $b_0 \neq b'_0$ implies $b_0 \cap b'_0 = \varnothing$. Thus in our model we dispensed with the possibility to construct finer or coarser registration procedures relative to a given one. Correspondingly, no direct mixtures of recording methods are considered which, however, could easily be provided analogous to direct mixtures of preparing procedures. This feature is certainly not relevant for the structure of quantum mechanics.

$\mathbf{R}$ is a system of section procedures such that the sets $\mathbf{R}(b_0) = \{b \in \mathbf{R}/b \subset b_0\}$ are Boolean algebras for all $b_0 \in \mathbf{R}_0$ and $b_0 \cap b'_0 = \varnothing$ for $b_0 \neq b'_0$. Moreover, $\mathbf{R}_0, \mathbf{R}$ fulfill axiom APS 4 of Ludwig (1977). Hence axioms APS 1, ..., 4 are satisfied.

In addition, lemma 1 shows that all observables can be measured approximately in the following sense: For a finite observable $F: \Sigma \to L$ and a σ^*-neighborhood U of 0 there is a recording method $b_0 \in R_0$ and an isomorphism $i: \Sigma \to R(b_0)$ such that $[F(\sigma) - \psi(b_0, i(\sigma))] \in U$ for all $\sigma \in \Sigma$.

The set of microsystems prepared by $a_{T,\tau} \in \mathbf{Q}$ and recorded by the method b_{0S} is

$$a_{T,\tau} \cap b_{0,S} = \bigcup_{i \in \tau} [\{(T, i, S)\} \times \sigma^*(\nu_i, S)]$$

$a_{T,\tau} \cap b_{0S}$ is not empty if $a_{T,\tau} \neq \varnothing$ and $b_{0S} \neq \varnothing$ [Axiom APS 5 of Ludwig (1977)].

By the definition

$$\mu(a_{T,\tau}, (b_{0S}, b_{S\sigma})) = \operatorname{Tr}(\varphi(a_{T,\tau}), \psi(b_{0S}, b_{S\sigma})) = \operatorname{Tr}(W(T, \tau)\cdot F(S, \sigma))$$

a function μ is defined on $\mathbf{Q}' \times \mathscr{F}$ ($\mathbf{Q}' = \mathbf{Q}\backslash\{\varnothing\}$) such that the conditions (1), (4), and (5) of Ludwig (1977), p. 199, are obviously satisfied. Condition (7) reads $a_{T,\tau} \cap b_{S,\sigma} = \varnothing$ is equivalent to

$$\mu(a_{T,\tau}, (b_{0S}, b_{S\sigma})) = 0 \qquad \text{for} \qquad a_{T,\tau} \in \mathbf{Q}' \text{ and } (b_{0S}, b_{S\sigma}) \in \mathscr{F}$$

Since

$$a_{T,\tau} \cap b_{S\sigma} = \bigcup_{i\in\tau} [\{(T, i, S)\} \times (\sigma \cap \sigma^*(i, S))]$$

we have

$$\begin{aligned} a_{T,\tau} \cap b_{S\sigma} = \varnothing \quad & \text{iff } \sigma \cap \sigma^*(\nu_i, S) = \varnothing \text{ for all } i \in \tau \\ & \text{iff } \operatorname{Tr}(W_{\nu_i}\cdot F(S, \sigma)) = 0 \text{ for all } i \in \tau \\ & \text{iff } \left(\sum_{i\in\tau} \lambda_i\right) \cdot \sum_{i\in\tau} \lambda_i \operatorname{Tr}(W_{\nu_i}\cdot F(S, \sigma)) = 0 \end{aligned}$$

[Thus the introduction of $\sigma^*(\nu, S)$ in (2.3) assures that the set of microsystems prepared by a procedure corresponding to W_ν and recorded by a procedure corresponding to $F(S, \sigma)$ is indeed empty if $\operatorname{Tr}(W_\nu \cdot F(S, \sigma)) = 0$.]

According to the theorem quoted by Ludwig (1977) on p. 199, conditions (1), (4), (5), and (7) for the function μ imply that axioms APS 6 and 7 are fulfilled.

In an axiomatic foundation of quantum mechanics a dual pair of Banach spaces B, B' can be constructed starting from a set M endowed with a structure ($\mathbf{Q}$, $\mathbf{R}_0$, $\mathbf{R}$) satisfying the axioms APS 1, . . . , 7.

The dual pairing is unique up to isomorphism [cf. Neumann (1972)]. Applied to the set M of our model this construction leads back to the Banach spaces B and B' spanned by the sets K and L, respectively, which were the starting point of the development of the model. This result holds since $\varphi(\mathbf{Q}')$ and $\psi(\mathscr{F})$ are dense in K and L, respectively.

Summarizing the discussion, we have shown that the set of axioms concerning the preparing and recording of microsystems given by Ludwig is free of contradictions and admits of nontrivial solutions.

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APPENDIX

Proof of lemma 1: Since H is separable it is easily shown that there is a countable subset $H_c \subset H$ such that for a finite orthonormal subset $\{\varphi_i\}_{i=1,\ldots,n} \subset H$ there is an orthonormal subset $\{\varphi_i'\}_{i=1,\ldots,n} \subset H_c$ with $\|\varphi_i - \varphi_i'\| \leqslant \epsilon$. Let C_c be a countable dense set of complex numbers. Let

$$\Omega = \left\{\varphi = \left|\sum_{i=1}^{n} \alpha_i\varphi_i\right|^{-1} \cdot \sum_{i=1}^{n} \alpha_i\varphi_i / \alpha_i \in C_c,\ \varphi_i \in H_c\right\}$$

$$L_c' = \left\{F = \sum_{i=1}^{n} \lambda_i P_{\psi_i} / \lambda_i \text{ rational},\ 0 \leqslant \lambda_i \leqslant 1,\ \psi_i \in \Omega \text{ pairwise orthogonal}\right\}$$

$$L_c'' = \left\{F = 1 - \sum_{i=1}^{n} F_i / F_i \in L_c',\ \sum_{i=1}^{n} F_i \leqslant 1\right\}$$

$L_c = L_c' \cup L_c''$ is denumerable and satisfies the conditions stated in the lemma. To prove this assertion consider a set $S \in M_0$, i.e., $S = \{F_i\}_{i=1,\ldots,n}$, $F_i \in L$, $F_i \neq 0$, $\sum_{i=1}^{n} F_i = 1$, and a σ^*-neighborhood U of $0 \in B(H)$. Restricted to L the neighborhood U can be characterized by $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{\tilde{m}} \in H$, $\epsilon > 0$. Let $\varphi_1, \ldots, \varphi_m$ be an orthonormal base of the subspace H_U of H spanned by $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_{\tilde{m}}$ and let P be the corresponding orthogonal projection. We have $PF_iP = \sum_{\rho=1}^{\kappa_i} \lambda_{i\rho} P_{\psi_{i\rho}}$ with $0 \leqslant \lambda_{i\rho} \leqslant 1$ and $\{\psi_{i\rho}\}_{\rho=1,\ldots,\kappa_i} \subset H_U$. There is an orthonormal set $\{\varphi_i'\}_{i=1,\ldots,m} \subset H_c$ with $\|\varphi_i' - \varphi_i\| \leqslant \epsilon_1$. By means of the components of the $\varphi_{i\rho}$ with respect to the base $\{\varphi_i\}_{i=1,\ldots,m}$, one defines $\psi_{i\rho}' \in \Omega$ with $\|\psi_{i\rho} - \psi_{i\rho}'\| \leqslant m(\epsilon_1 + 3\epsilon_2)$, ϵ_2 being a constant independent of ϵ_1 caused by the fact that C_c is dense in C. Hence

$$\|P_{\psi_{i\rho}} - P_{\psi_{i\rho}'}\| \leqslant 2m(\epsilon_1 + 3\epsilon_2)$$

For $i = 1, \ldots, n - 1$ we define $F_i' = \sum_{\rho=1}^{k_i} \lambda_{i\rho}' P_{\psi_{i\rho}'}$ with $\lambda_{i\rho}'$ rational and

$$\lambda_{i\rho}(1 - \epsilon_3) \leqslant \lambda_{i\rho}' \leqslant \lambda_{i\rho}(1 - \epsilon_3/2) \qquad \epsilon_3 > 0$$

and $F_n' = 1 - \sum_{i=1}^{n-1} F_i'$.

Since the following estimation shows $\sum_{i=1}^{n-1} F_i' \leqslant 1$, we have $F_i' \in L_c'$ for $i = 1, \ldots, n - 1$ and $F_n' \in L_c''$. Considering $\varphi' = \sum_{i=1}^{m} \alpha_i\varphi_i'$, one obtains

$$\left\langle \varphi' \middle| \sum_{i=1}^{n-1} F_i'\varphi' \right\rangle \leqslant (1 - \epsilon_3/2)\left\langle \varphi \middle| \sum_{i=1}^{n-1} F_i\varphi \right\rangle + 2nm^3\epsilon_2$$
$$\leqslant 1 - \epsilon_3/2 + 2nm^3\epsilon_2$$

with $\varphi = \sum_{i=1}^{m} \alpha_i\varphi_i$.

Moreover, $\|F_i' - PFP\| \leqslant m\epsilon_3 + 2m^2(\epsilon_1 + 3\epsilon_2)$ holds, and hence $F_i - F_i' \in U_{\tilde{\varphi}_1,\ldots,\tilde{\varphi}_{\tilde{m}},\varepsilon}$ for appropriately chosen $\epsilon_1, \epsilon_2, \epsilon_3$.

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